

t -ANALOGS OF q -CHARACTERS OF QUANTUM AFFINE ALGEBRAS OF TYPE E_6, E_7, E_8

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ABSTRACT. We compute t -analogs of q -characters of all l -fundamental representations of the quantum affine algebras of type $E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$ by a supercomputer. In particular, we prove the fermionic formula for Kirillov-Reshetikhin modules conjectured by Hatayama et al. [6] for these classes of representations. We also give explicitly the monomial realization of the crystal of the corresponding fundamental representations of the quantum enveloping algebras associated with finite dimensional Lie algebras of types E_6, E_7, E_8 . These are computations of Betti numbers of graded quiver varieties, quiver varieties and determination of all irreducible components of the lagrangian subvarieties of quiver varieties of types E_6, E_7, E_8 respectively.

INTRODUCTION

Let \mathfrak{g} be a simple Lie algebra of type ADE over \mathbb{C} with the index set I of simple roots, $\mathbf{L}\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}]$ be its loop algebra, and $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$ be its quantum universal enveloping algebra, or the quantum loop algebra for short. It is a subquotient of the quantum affine algebra $\mathbf{U}_q(\widehat{\mathfrak{g}})$, i.e. without central extension and degree operator. It contains the quantum enveloping algebra $\mathbf{U}_q(\mathfrak{g})$ associated with \mathfrak{g} as a subalgebra.

By Drinfeld [3] and Chari-Pressley [2], simple $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$ -modules are parametrized by I -tuples of polynomials $P = (P_i(u))_{i \in I}$ with normalization $P_i(0) = 1$. They are called *Drinfeld polynomials*. Let us denote by $L(P)$ the simple module with Drinfeld polynomial P . When P is given by $P_i(u) = (1 - au)^{\delta_{iN}}$ for a given $N \in I$, we call corresponding module an N^{th} l -fundamental representation. (It has been called a *level 0 fundamental module* or simply *fundamental representation* in some literature.) We can assume $a = 1$ without the loss of generality as the general module is a pullback of the module with $a = 1$ by an algebra automorphism of $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$.

Let $\chi_{q,t}(L(P))$ be the t -analog of q -character of a simple module $L(P)$ defined by the author [14, 15]. It is defined via the geometry of graded quiver varieties. It values in certain Laurent polynomial ring with infinitely many variables with integer coefficients. It is a t -analog of the q -character $\chi_q(L(P))$ introduced earlier [12, 5], which was a refinement of the ordinary character of the restriction of $L(P)$ to a $\mathbf{U}_q(\mathfrak{g})$ -module. In [14, 15] we “computed” $\chi_{q,t}(L(P))$ for arbitrary given $L(P)$, in the sense that we gave a purely combinatorial algorithm to write down all monomials and coefficients in $\chi_{q,t}(L(P))$ where the final expression involves only $+$, \times , integers and variables.

In order to clarify in what sense our result is new compared with earlier results, we *define* what the word compute mean precisely. When we write the word compute in the quotation marks, it means that we give a combinatorial algorithm to compute something in the above sense. It does not necessarily mean that we actually compute it. We can write a computer program in principle, but the question whether we can actually compute it or not depends on the size of computer memory. (For example, it is clear that the rank n of \mathfrak{g} cannot be

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larger than the size of the memory.) On the other hand, when we write the word compute without the quotation mark, we mean to compute something in a strict sense, i.e. we express something so that it contains only finitely many \pm , \times , integers and variables. For example, if we write $x = \sum_{i=1}^{2^{(2^{100})}} a_i$ for some explicit a_i , we “compute” x , but we do not compute x unless we actually compute the sum. On the other hand, we do not require that the final expression can be read by the human, as such a concept cannot make precise.

The algorithm is separated into three steps:

- (1) “Computation” of $\chi_{q,t}$ for l -fundamental representations.
- (2) “Computation” of $\chi_{q,t}$ for standard modules, i.e. tensor products of l -fundamental representations.
- (3) “Computation” of the t -analog of the composition factors of simple modules in standard modules.

The third step is analogous to the definition of Kazhdan-Lusztig basis. If $M(P)$ denote the standard module, we have

$$(0.1) \quad \overline{\chi_{q,t}(L(P))} = \chi_{q,t}(L(P)), \quad \chi_{q,t}(L(P)) = \chi_{q,t}(M(P)) + \sum_{Q: Q < P} a_{PQ}(t) \chi_{q,t}(M(Q))$$

for some $a_{PQ}(t) \in t^{-1}\mathbb{Z}[t^{-1}]$, where ‘ $<$ ’ is a certain explicitly defined ordering. Thus $a_{PQ}(t)$ is analogous to Kazhdan-Lusztig polynomials. The above characterization allows us to “compute” $a_{PQ}(t)$, once $\chi_{q,t}(M(P))$ is “computed”. (And it is known that the actual computation of Kazhdan-Lusztig polynomials is very hard.)

In the second step, we express $\chi_{q,t}(M(P))$ as a twisted multiplication of $\chi_{q,t}$ of l -fundamental representations. It is almost the same as usual multiplication on the polynomials, but a product of two monomials m, m' is twisted as $t^{2d(m,m')}mm'$. Therefore this step is very simple. It is clear that $\chi_{q,t}(M(P))$ can be “computed” if $\chi_{q,t}$ of l -fundamental representations are “computed”.

This paper concerns the first step. Our “computation” in [14, 15] was t -analog of the “computation” by Frenkel-Mukhin [4]. It is based on the observation that (a) $\chi_{q,t}$ satisfies a certain analog of the Weyl group invariance of the ordinary characters, and (b) the l -fundamental representation satisfies a certain property analogous to that of minuscule representations of \mathfrak{g} . Recall that a simple finite dimensional representation of \mathfrak{g} is called *minuscule* if all weights are conjugates of the highest weight under the Weyl group each occurring with multiplicity 1.

When \mathfrak{g} is of classical type, i.e. of type A, D , the author gave a tableaux sum expression of $\chi_{q,t}$ of l -fundamental representations [16]. It means that we give another “computation” of $\chi_{q,t}$, which are more familiar to us than the above one. It does not mean we compute $\chi_{q,t}$ in our strict sense. In fact, the comparison of two methods does not make sense unless we define what we mean by ‘familiar’. In practice, it just means that we have a faster algorithm for the actual computer calculation.

In this paper we report the actual computer computation of $\chi_{q,t}$ of l -fundamental representations when \mathfrak{g} is of type $E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$. Our algorithm is implemented in the computer language C. The source code is available at <http://www.math.kyoto-u.ac.jp/~nakajima/Qchar/>. The author’s personal computer (Dell Dimension 9100) can give the answer up to the 6th l -fundamental representation of E_8 , where our numbering of I is the following:

$$\begin{array}{cccccccc} 7 & - & 6 & - & 5 & - & 4 & - & 3 & - & 2 & - & 1 \\ & & & & | & & & & & & & & \\ & & & & 8 & & & & & & & & \end{array}$$

We need about 120Mbytes of the memory for this calculation. For the 4th and 5th l -fundamental representations, the computation was done on a supercomputer FUJITSU HPC 2500 at Kyoto University. The calculation required about 2.6Gbytes (for 4th) and 120Gbytes (for 5th) of memory, and it took 6 hours and 350 hours for the calculation respectively. The final answers (stored in a compressed format as explained below) are 3.2Gbytes and 180Gbytes respectively. In fact, the calculation of the 4th one was done several years ago and was mentioned in some of the author's papers. However we needed to wait for the Kyoto University to renovate the supercomputer so that we can use 120Gbytes of memory in a single program, and then wait for the author to get an enough budget to use the supercomputer.

As far as the author knows, the computation (in our strict sense) for the 5th one was not known before. Frenkel-Mukhin, Hernandez-Schedler told the author that they wrote computer programs calculating $\chi_{q,t=1}$ and $\chi_{q,t}$ respectively. But both had a problem of computer memory.

In conclusion, we can now delete the quotation mark for computation in the first step of the algorithm for type E above.

As an application, we can compute t -analog of the ordinary characters of the restrictions of l -fundamental representations to $\mathbf{U}_q(\mathfrak{g})$ -modules. The l -fundamental modules are examples of the so-called Kirillov-Reshetikhin modules. Kirillov-Reshetikhin gave conjectural formula for the ordinary character of the restriction of a Kirillov-Reshetikhin module [10]. Its graded version (i.e. t -analog) together with an interpretation in terms of the conjectural crystal base was given by Hatayama, Kuniba, Okado, Takagi and Yamada [6]. Then Lusztig conjectured that their conjectural grading is the same as the cohomological degree [13], in a certain class of Kirillov-Reshetikhin modules including l -fundamental representations. Therefore the formula in [6], in the class, gives the generating function of Poincaré polynomials of quiver varieties. In general, the conjectural formula is expressed as a summation over partitions, and called a *fermionic formula*. The author gave an expression for $t = 1$ in [17, Cor. 1.3] (the result was extended to type $BCFG$ in [7]). It is again given as a summation over partition, but the definition of the binomial coefficient appearing in the coefficients is different. The equivalence between two expressions are not known so far, therefore the original fermionic formula is remained open.

For an l -fundamental representation, the original fermionic formula can be given by an explicit polynomial by the so-called Kleber's algorithm [11]. Here we do not make precise what we mean by 'explicit'. For types A, D , it was shown in [16] that this 'explicit' expression for an l -fundamental representation is equal to the "computation" in [14]. For type E , the algorithm can be used to compute the fermionic formula in our strict sense. Then the result can be checked in some special cases previously computed (at least for $t = 1$) (e.g. [1]), but most of l -fundamental representations have remained open. Remark that Kleber's algorithm does not apply to the modified formula in [17], so it is not known that the modified formula gives the computation in the strict sense.

Our computation of $\chi_{q,t}$ gives the explicit expression and we find that it is the same as one given in [6]. Therefore we prove Lusztig's conjecture for all l -fundamental representations.

Also as another application, we determine all monomials appearing in the monomial realization of the crystal corresponding to fundamental representations of type E . For types A, D , they were determined in [16] as an application of the explicit description of $\chi_{q,t}$ of l -fundamental representations. For types B, C , they were determined in [9]. For types F, G , they can be easily determined (cf. [8]). In conclusion, we describe the monomial realization of the crystals of all fundamental representations explicitly.

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1. t -ANALOGS OF q -CHARACTERS

We shall not give the definition of quantum loop algebras, nor their finite dimensional representations in this paper. (See [14] for a survey.) We just review properties of $\chi_{q,t}$, as axiomized in [15].

Let $\mathcal{Y}_t \stackrel{\text{def.}}{=} \mathbb{Z}[t, t^{-1}, Y_{i,a}, Y_{i,a}^{-1}]_{i \in I, a \in \mathbb{C}^*}$ be a Laurent polynomial ring of uncountably many variables $Y_{i,a}$'s with coefficients in $\mathbb{Z}[t, t^{-1}]$. A *monomial* in \mathcal{Y}_t means a monomial only in $Y_{i,a}^\pm$, containing no t 's. Therefore a polynomial is a sum of monomials multiplied by Laurent polynomials in t , called coefficients as usual. Let

$$A_{i,a} \stackrel{\text{def.}}{=} Y_{i,a\varepsilon} Y_{i,a\varepsilon^{-1}} \prod_{j:j \neq i} Y_{j,a}^{c_{ij}},$$

where c_{ij} is the (i, j) -entry of the Cartan matrix. Let \mathcal{M} be the set of monomials in \mathcal{Y}_t .

Definition 1.1. (1) For a monomial $m \in \mathcal{M}$, we define $u_{i,a}(m) \in \mathbb{Z}$ be the degree in $Y_{i,a}$, i.e.

$$m = \prod_{i,a} Y_{i,a}^{u_{i,a}(m)}.$$

(2) A monomial $m \in \mathcal{M}$ is said *i -dominant* if $u_{i,a}(m) \geq 0$ for all a . It is said *l -dominant* if it is i -dominant for all i .

(3) Let m, m' be monomials in \mathcal{M} . We say $m \leq m'$ if m/m' is a monomial in $A_{i,a}^{-1}$ ($i \in I, a \in \mathbb{C}^*$). Here a monomial in $A_{i,a}^{-1}$ means a product of nonnegative powers of $A_{i,a}^{-1}$. It does not contain any factors $A_{i,a}$. In such a case we define $v_{i,a}(m, m') \in \mathbb{Z}_{\geq 0}$ by

$$m = m' \prod_{i,a} A_{i,a}^{-v_{i,a}(m, m')}.$$

This is well-defined since the ε -analog of the Cartan matrix is invertible. We say $m < m'$ if $m \leq m'$ and $m \neq m'$.

(4) For an i -dominant monomial $m \in \mathcal{M}$ we define

$$E_i(m) \stackrel{\text{def.}}{=} m \prod_a \sum_{r_a=0}^{u_{i,a}(m)} t^{r_a(u_{i,a}(m)-r_a)} \begin{bmatrix} u_{i,a}(m) \\ r_a \end{bmatrix}_t A_{i,a\varepsilon}^{-r_a},$$

where $\begin{bmatrix} n \\ r \end{bmatrix}_t$ is the t -binomial coefficient.

(5) We define a ring involution $\overline{}$ on \mathcal{Y}_t by $\bar{t} = t^{-1}$, $\overline{Y_{i,a}^\pm} = Y_{i,a}^\pm$.

Suppose that l -dominant monomials m_{P^1} , m_{P^2} and monomials $m^1 \leq m_{P^1}$, $m^2 \leq m_{P^2}$ are given. We define an integer $d(m^1, m_{P^1}; m^2, m_{P^2})$ by

$$(1.2) \quad d(m^1, m_{P^1}; m^2, m_{P^2}) \stackrel{\text{def.}}{=} \sum_{i,a} (v_{i,a\varepsilon}(m^1, m_{P^1}) u_{i,a}(m^2) + u_{i,a\varepsilon}(m_{P^1}) v_{i,a}(m^2, m_{P^2})).$$

For an I -tuple of rational functions $Q/R = (Q_i(u)/R_i(u))_{i \in I}$ with $Q_i(0) = R_i(0) = 1$, we set

$$m_{Q/R} \stackrel{\text{def.}}{=} \prod_{i \in I} \prod_{\alpha} \prod_{\beta} Y_{i,\alpha} Y_{i,\beta}^{-1},$$

where α (resp. β) runs roots of $Q_i(1/u) = 0$ (resp. $R_i(1/u) = 0$), i.e. $Q_i(u) = \prod_{\alpha} (1 - \alpha u)$ (resp. $R_i(u) = \prod_{\beta} (1 - \beta u)$). As a special case, an I -tuple of polynomials $P = (P_i(u))_{i \in I}$ defines $m_P = m_{P/1}$. The l -dominant monomial m_{P^α} appeared above is associated to an I -tuple of polynomials $P = (P_i(u))_{i \in I}$. In this way, the set \mathcal{M} of monomials are identified with the set of I -tuple of rational functions, and the set of l -dominant monomials are identified with the set of I -tuple of polynomials.

The t -analog of the Grothendieck ring \mathbf{R}_t is a free $\mathbb{Z}[t, t^{-1}]$ -module with base $\{M(P)\}$ where $P = (P_i(u))_{i \in I}$ is the Drinfeld polynomial. (We do not recall the definition of standard modules $M(P)$ here, but the reader safely consider them as formal variables.)

The t -analog of the ε -character homomorphism is a $\mathbb{Z}[t, t^{-1}]$ -linear homomorphism $\chi_{q,t}: \mathbf{R}_t \rightarrow \mathcal{Y}_t$. It is defined as the generating function of Poincaré polynomials of graded quiver varieties, or the generating function of graded dimensions of l -weight spaces of a $\mathbf{U}_q(\mathbf{Lg})$ -module [18], and will not be reviewed in this paper.

We also need a slightly modified version:

$$\widetilde{\chi}_{q,t}(M(P)) = \sum_m t^{d(m, m_P; m, m_P)} a_m(t) m \quad \text{if } \chi_{q,t}(M(P)) = \sum_m a_m(t) m.$$

If we know one of $\chi_{q,t}$ and $\widetilde{\chi}_{q,t}$, we know the remaining one.

The following was proved in [14, 15]:

Fact 1.3. (1) *The $\chi_{q,t}$ of a standard module $M(P)$ has a form*

$$\chi_{q,t}(M(P)) = m_P + \sum a_m(t) m,$$

where the summation runs over monomials $m < m_P$.

(2) *For each $i \in I$, $\widetilde{\chi}_{q,t}(M(P))$ can be expressed as a linear combination (over $\mathbb{Z}[t, t^{-1}]$) of $E_i(m)$ with i -dominant monomials m .*

(3) *Suppose that two I -tuples of polynomials $P^1 = (P_i^1)$, $P^2 = (P_i^2)$ satisfy the following condition:*

$$(1.4) \quad \begin{aligned} & a/b \notin \{\varepsilon^n \mid n \in \mathbb{Z}, n \geq 2\} \text{ for any pair } a, b \text{ with } P_i^1(1/a) = 0, \\ & P_j^2(1/b) = 0 \text{ (} i, j \in I \text{)}. \end{aligned}$$

Then we have

$$\widetilde{\chi}_{q,t}(M(P^1 P^2)) = \sum_{m^1, m^2} t^{2d(m^1, m_{P^1}; m^2, m_{P^2})} a_{m^1}(t) a_{m^2}(t) m^1 m^2,$$

where $\widetilde{\chi}_{q,t}(M(P^a)) = \sum_{m^a} a_{m^a}(t) m^a$ with $a = 1, 2$.

Moreover, properties (1), (2), (3) uniquely determine $\chi_{q,t}(M(P))$.

(4) *The $\chi_{q,t}$ of the simple module $L(P)$ is given by (0.1).*

Apart from the existence problem, one can consider the above properties (1), (2), (3) as the definition of $\chi_{q,t}$ (an axiomatic definition). We only use the above properties, and the reader can safely forget the original definition. Note that we will prove the existence of $\chi_{q,t}$ by our computer calculation.

By the property (1) we call the monomial m_P corresponding to the Drinfeld polynomial P l -highest weight monomial.

2. ALGORITHM

In this section we shall explain our algorithm to determine $\widetilde{\chi_{q,t}}(L(P))$ recursively starting from the l -dominant weight monomial m_P . It is a slight modification of one in [4]. We shall also explain why we require large memory to compute $\chi_{q,t}$ of the 5th l -fundamental representation of $\mathbf{U}_q(\mathbf{Lg})$ with $\mathbf{g} = E_8$. The problem does not exist for the other l -fundamental representations.

We take a Drinfeld polynomial $P = (P_i(u))$ $P_i(u) = (1 - u)^{\delta_{iN}}$ corresponding to the N^{th} l -fundamental representation.

One of the key property of $\chi_{q,t}$ of an l -fundamental representation is that all monomials appearing in $\chi_{q,t}$ are not l -dominant except the l -highest one. This was proved in [4, Cor. 4.5] and [15, 4.13].

For each monomial m in $\widetilde{\chi_{q,t}}(L(P))$ we determine the coefficient $a_m(t) \in \mathbb{Z}[t]$ and the I -tuple of polynomial $(a_{m,i}(t))_{i \in I} \in \mathbb{Z}[t]$ (called *coloring*) recursively. Let us introduce several concepts. We say m is *admissible* if all $a_{m,i}(t)$ are the same for any i such that m is not i -dominant. We say *the algorithm fails at m* if m is not admissible. We say *the algorithm stops at m* if m is l -dominant.

Now we explain the algorithm. At the first stage we set $a_{m_P}(t) = 1$ and $a_{m_P,i}(t) = 0$ for all $i \in I$ for the l -highest weight monomial m_P . Next take a monomial m such that $a_m(t)$ and $a_{m,i}(t)$ are determined. If m is not i -dominant for any i (this will happen if m the l -lowest weight vector), we do nothing on m and go to the next monomial. If m is i -dominant, we compute $(a_m(t) - a_{m,i}(t))E_i(m)$. We call this procedure the *i -expansion at m* . We add a monomial m' appearing there to the list. And for a monomial m' in the list, we set $a_{m',i}(t)$ be the sum of the contribution to m' in the i -expansion at m for various $m < m'$ which is i -dominant. As there is only finitely many $m < m'$, $a_{m',i}(t)$ will be eventually determined. After all $a_{m',i}(t)$ are determined in this way, we can ask m' is admissible or not. If m' is not admissible (i.e. the algorithm fails at m'), we stop. If m' is l -dominant (i.e. the algorithm stop at m'), we stop. If m' is admissible and not l -dominant, we set $a_{m'}(t) = a_{m',i}(t)$ for some (and any by admissibility) i such that m' is not i -dominant. We continue this procedure until all $a_m(t)$ and $a_{m,i}(t)$ are determined, and all $(a_m(t) - a_{m,i}(t))E_i(m)$ are expanded, or we stop at some m .

Now we apply the algorithm starting from the l -highest weight monomial m_P . As $\widetilde{\chi_{q,t}}(L(P))$ satisfies the properties (1),(2) in Fact 1.3, the algorithm cannot fail. As $\widetilde{\chi_{q,t}}(L(P))$ does not contain l -dominant monomials other than l -highest one, the algorithm cannot stop. Finally as $L(P)$ is a finite dimensional, $\widetilde{\chi_{q,t}}(L(P))$ contains only finitely many monomials. Therefore we eventually determine all $a_m(t)$ and $a_{m,i}(t)$.

Remark 2.1. If we apply the same algorithm in case \mathbf{g} is a Kac-Moody Lie algebra (say an affine Lie algebra), the algorithm does not fail, does not stop, but we always get a new monomial in the expansion. Therefore the procedure never end.

Now we consider the 5th l -fundamental representation of $\mathbf{U}_q(\mathbf{Lg})$ with $\mathbf{g} = E_8$ and we will explain the reason why we need various tricks to save the size of data. Because of these tricks, we had not known how big the total size is in advance, so we used the following guess: We know that the dimension of the 4th fundamental representation of \mathbf{g} is 146325270, while 5th one is 6899079264. Therefore we expect that the corresponding $\chi_{q,t}$'s have a similar ratio. We first compute the 4th l -fundamental representations and expect that the total size of the 5th one is about 50 times as much. This turned out to be approximately correct as we can see from the data in Introduction.

By [16, Prop. 3.4] the set of monomials appearing in the q -character of an l -fundamental representation has a $\mathbf{U}_q(\mathfrak{g})$ -crystal structure, which is isomorphic to the corresponding fundamental representation of $\mathbf{U}_q(\mathfrak{g})$. In particular, the number of the monomials appearing in the 5th l -fundamental representation is equal to the dimension of the 5th fundamental representation of $\mathfrak{g} = E_8$, i.e. $6899079264 \approx 6.4 \times 2^{30} = 6.4\text{Giga}$. For each monomial m , we must remember (a) the expression of the monomial and (b) the coloring, i.e. an I -tuple of polynomials in t .

Let us first consider how we can express the monomial. It is known that l -lowest weight monomial, i.e. the unique monomial with (ordinary) weight $-\varpi_5$, is $Y_{5,q^{30}}^{-1}$ (see e.g. [4, 6.8]). We have

$$\begin{aligned} Y_{5,q^{30}}^{-1} = & Y_{5,1} \times A_{1,q^5} A_{1,q^7} A_{1,q^9} A_{1,q^{11}} A_{1,q^{13}} A_{1,q^{15}}^2 A_{1,q^{17}} A_{1,q^{19}} A_{1,q^{21}} A_{1,q^{23}} A_{1,q^{25}} \\ & \times A_{2,q^4} A_{2,q^6}^2 A_{2,q^8}^2 A_{2,q^{10}}^2 A_{2,q^{12}}^2 A_{2,q^{14}}^3 A_{2,q^{16}}^2 A_{2,q^{18}}^2 A_{2,q^{20}}^2 A_{2,q^{22}}^2 A_{2,q^{24}}^2 A_{2,q^{26}} \\ & \times A_{3,q^3} A_{3,q^5}^2 A_{3,q^7}^3 A_{3,q^9}^3 A_{3,q^{11}}^3 A_{3,q^{13}}^4 A_{3,q^{15}}^4 A_{3,q^{17}}^4 A_{3,q^{19}}^3 A_{3,q^{21}}^3 A_{3,q^{23}}^3 A_{3,q^{25}}^2 A_{3,q^{27}} \\ & \times A_{4,q^2} A_{4,q^4}^2 A_{4,q^6}^3 A_{4,q^8}^4 A_{4,q^{10}}^4 A_{4,q^{12}}^5 A_{4,q^{14}}^5 A_{4,q^{16}}^5 A_{4,q^{18}}^4 A_{4,q^{20}}^4 A_{4,q^{22}}^4 A_{4,q^{24}}^3 A_{4,q^{26}}^2 A_{4,q^{28}} \\ & \times A_{5,q^1} A_{5,q^3}^2 A_{5,q^5}^3 A_{5,q^7}^4 A_{5,q^9}^5 A_{5,q^{11}}^6 A_{5,q^{13}}^6 A_{5,q^{15}}^6 A_{5,q^{17}}^5 A_{5,q^{19}}^6 A_{5,q^{21}}^5 A_{5,q^{23}}^4 A_{5,q^{25}}^3 A_{5,q^{27}}^2 A_{5,q^{29}} \\ & \times A_{6,q^2} A_{6,q^4}^2 A_{6,q^6}^2 A_{6,q^8}^3 A_{6,q^{10}}^4 A_{6,q^{12}}^4 A_{6,q^{14}}^4 A_{6,q^{16}}^4 A_{6,q^{18}}^4 A_{6,q^{20}}^3 A_{6,q^{22}}^2 A_{6,q^{24}}^2 A_{6,q^{26}}^2 A_{6,q^{28}} \\ & \times A_{7,q^3} A_{7,q^5} A_{7,q^7} A_{7,q^9}^2 A_{7,q^{11}}^2 A_{7,q^{13}}^2 A_{7,q^{15}}^2 A_{7,q^{17}}^2 A_{7,q^{19}}^2 A_{7,q^{21}}^2 A_{7,q^{23}} A_{7,q^{25}} A_{7,q^{27}} \\ & \times A_{8,q^2} A_{8,q^4} A_{8,q^6}^2 A_{8,q^8}^2 A_{8,q^{10}}^3 A_{8,q^{12}}^3 A_{8,q^{14}}^3 A_{8,q^{16}}^3 A_{8,q^{18}}^3 A_{8,q^{20}}^2 A_{8,q^{22}}^2 A_{8,q^{24}}^2 A_{8,q^{26}} A_{8,q^{28}}. \end{aligned}$$

Any other monomial is given equal to $Y_{5,1}$ multiplied by a part of A_{i,q^k} 's appeared above. We record the monomial as a sequence of A_{i,q^k}^m 's, where i runs 1 to 8, k runs from 1 to 29, and m runs from 1 to 6. We can store the triple (i, k, m) in a single **short int**, i.e. 16bit of memory. The length of the sequence is at most 106, which is the length for $Y_{5,q^{30}}^{-1}$. A naive count gives $6899079264 \times 106 \times 16\text{bit} > 1300\text{Gbyte}$. This is too large. Therefore we use the following trick: Noticing that many monomials share the same sequences of A_{i,q^k}^m 's, we store the data into a tree so that we do not need to repeat the common part. By this trick, it becomes uncertain how much size we need in advance, as we mentioned above.

Next let us turn to coloring. By [15], $\chi_{q,t}(L(P)) = \sum_m a_m(t)m$ is given by the Poincaré polynomials of various graded quiver varieties corresponding to m . Therefore the degree of the coefficient $a_m(t)$ is equal to the (real) dimension of the variety corresponding to m . On the other hand, the dimension of the graded quiver variety is bounded by the half of the ordinary quiver variety containing it. For the 5th fundamental representation, the maximum (among various connected components) of the dimension is equal to 60. Therefore the maximum of the degree is 30. As $a_{m,i}(t)$ is given by a virtual Hodge polynomial of a certain stratum of the graded quiver variety, the degree is also less than or equal to 30. As $a_m(t)$, $a_{m,i}(t)$ are polynomials in t^2 , we have $30/2 + 1 = 16$ coefficients. Therefore we must record 16×8 integers for each monomial. We did not know how large integers were in advance. As a result of our calculation, it turns out we can store it into a **short int**. Then we would need $16 \times 8 \times 16\text{bit} = 256\text{byte}$ for each monomial. This is huge size, though it could be handled by our computer probably. However we note that many monomials m have coefficient $a_m(t) = 1$. We store $a_{m,i}(t)$ for those monomials in a special format to save the size of data. As we do not need $a_{m,i}(t)$ for the final result, they are not included. (As a result of our calculation we find 4639565354 among 6899079264 monomials have this property.)

We have explained the total size of the data so far. In practice, it is more important to know how much memory is required in the course of the calculation. For the simplicity of the program, we replace the ordering $<$ among monomials by more manageable ordering given by

$$\text{depth } m \stackrel{\text{def.}}{=} \sum_{i,a} v_{i,a}(m, m_P).$$

Therefore the l -highest weight vector has depth 0, $Y_{5,1}A_{5,q}^{-1}$ has depth 1, etc. We expand the monomial of depth 0, then monomials with depth 1, monomials with depth 2, and so on. When we expand all monomials of given depth, we store all obtained monomials together with coloring in memory. As a single monomial appears many times in the expansions at various monomials, it is not practical to save the data in the hard disk. Therefore the most crucial point is to save the size of data so that the program requires, in a fixed depth, up to 200Gbyte of memory, which is the limit of the supercomputer. We estimated the memory requirement by that for 4th l -fundamental representation as above, and we guessed that the calculation was possible. This turns out to be true fortunately.

3. RESULTS

We only consider the 5th l -fundamental representation of $\mathbf{U}_q(\mathbf{Lg})$ with $\mathbf{g} = E_8$.

As the final result is a huge polynomial, we cannot give it here. So we only give a part of the information. The monomial whose coefficient with the highest degree t^{30} is

$$\begin{aligned} & (1 + 4t^2 + 10t^4 + 20t^6 + 33t^8 + 47t^{10} + 59t^{12} + 66t^{14} \\ & \quad + 66t^{16} + 59t^{18} + 47t^{20} + 33t^{22} + 20t^{24} + 10t^{26} + 4t^{28} + t^{30}) \\ & \quad \times Y_{1,q^{14}}Y_{1,q^{16}}^{-1}Y_{3,q^{14}}^2Y_{3,q^{16}}^{-2}Y_{5,q^{14}}^3Y_{5,q^{16}}^{-3}Y_{7,q^{14}}Y_{7,q^{16}}^{-1}. \end{aligned}$$

The coefficient is the Poincaré polynomial of a certain graded quiver variety.

We define the t -graded character by

$$\text{ch}_t(L(P)) = \widetilde{\chi_{q,t}}(L(P))|_{Y_{i,a} \rightarrow y_i}.$$

If we put $t = 1$, it becomes the ordinary character of the restriction of $L(P)$ to $\mathbf{U}_q(\mathbf{g})$. It is also equal to the generating function of the Poincaré polynomials of the quiver varieties, where the degree 0 corresponding to the middle degree. For example, the coefficient of the weight 0 is

$$\begin{aligned} & 1357104 + 2232771t^2 + 2002423t^4 + 1317308t^6 + 716312t^8 + 342421t^{10} + 148512t^{12} \\ & \quad + 59490t^{14} + 22162t^{16} + 7687t^{18} + 2463t^{20} + 726t^{22} + 192t^{24} + 44t^{26} + 8t^{28} + t^{30}. \end{aligned}$$

Let $V(\lambda)$ denote the irreducible highest weight representation of $\mathbf{U}_q(\mathbf{g})$ with the highest weight λ . Let $\text{ch } V(\lambda)$ be its character. If we write

$$\text{ch}_t L(P) = \sum_{\lambda} M(P, \lambda, t) \text{ch } V(\lambda),$$

the coefficient $M(P, \lambda, t)$ is specialized to the multiplicity of $V(\lambda)$ in the restriction of $L(P)$ at $t = 1$. The fermionic formula mentioned in the Introduction is a conjectural expression of $M(P, \lambda, t)$ (for P corresponding to the Kirillov-Reshetikhin modules).

As we have computed $\widetilde{\chi_{q,t}}(L(P))$, $M(P, \lambda, t)$ can be given if we compute $V(\lambda)$. Let us compute $V(\lambda)$ by the method in [14, 7.1.1], i.e.

$$V(\lambda) = \widetilde{\chi_{q,t}}(L(Q))|_{Y_{i,a} \rightarrow y_i, t \rightarrow 0},$$

where Q corresponding to λ is given as follows: We choose an orientation for each edge of the Dynkin diagram and choose a function $m: I \rightarrow \mathbb{Z}$ such that $m(i) - m(j) = 1$ for an oriented edge $i \rightarrow j$. Then we take

$$Q_i(u) = (1 - uq^{m(i)})^{\langle \lambda, h_i \rangle}.$$

For this choice of Q , it is known that $\text{ch}_t(L(Q)) = \widetilde{\chi_{q,t}(L(Q))}|_{Y_{i,a} \rightarrow y_i}$ is equal to the generating function of shifted Poincaré polynomial of the quiver variety as above. In particular, it is independent of the choice of the orientation. For each dominant weight λ appearing in $\text{ch}_t L(P)$, we choose $Q = Q_\lambda$ as above and define matrices $P(t) = (P_{\lambda\mu}(t))$ and $IC(t) = (IC_{\lambda\mu}(t))$ by

$$\text{ch}_t L(Q_\lambda) = \sum_{\mu} P_{\lambda\mu}(t) e^\mu + \text{non dominant terms},$$

$$\text{ch}_t L(Q_\lambda) = \sum_{\mu} IC_{\lambda\mu}(t) \text{ch } V(\mu).$$

Then we have

$$IC(t) = P(t)P(0)^{-1}$$

By [14, 15] $IC_{\lambda\mu}(t)$ is the Poincaré polynomial of the stalk of the intersection cohomology sheaf of a stratum of the quiver variety corresponding to λ at a point in the stratum corresponding to μ . In our case it is given by

$$IC(t) = \left(\begin{array}{c|c|c} \text{Table 1} & \text{Table 2} & \text{Table 3} \\ \hline 0 & 0 & \text{Table 4} \end{array} \right),$$

where $y_i = e^{\varpi_i}$. The first row gives $\text{ch}_t(L(P))$ for the 5th l -fundamental representation $L(P)$. We see that it coincides with the conjectural formula in [6]. The same assertion for other l -fundamental representations can be proved by invoking other rows. The same can be proved for types E_6, E_7 in the same manner.

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TABLE 2.

	$\varpi_2 + \varpi_7$	$\varpi_1 + \varpi_8$	$2\varpi_7$	ϖ_6	$3\varpi_1$	$\varpi_1 + \varpi_2$	ϖ_3	$\varpi_1 + \varpi_7$
ϖ_5	$3t^4+4t^6+4t^8+2t^{10}+t^{12}$	$2t^4+5t^6+5t^8+3t^{10}+2t^{12}+t^{14}$	$3t^6+2t^8+3t^{10}+t^{12}+t^{14}$	$2t^4+4t^6+6t^8+4t^{10}+3t^{12}+t^{14}+t^{16}$	t^8+t^{12}	$2t^6+5t^8+5t^{10}+3t^{12}+2t^{14}+t^{16}$	$5t^6+5t^8+7t^{10}+4t^{12}+3t^{14}+t^{16}+t^{18}$	$2t^6+9t^8+10t^{10}+10t^{12}+6t^{14}+4t^{16}+2t^{18}+t^{20}$
$\varpi_3 + \varpi_7$	$t^2+3t^4+4t^6+2t^8+t^{10}$	$3t^4+5t^6+3t^8+2t^{10}+t^{12}$	$2t^4+2t^6+3t^8+t^{10}+t^{12}$	$2t^4+5t^6+4t^8+3t^{10}+t^{12}+t^{14}$	t^6+t^{10}	$t^4+4t^6+5t^8+3t^{10}+2t^{12}+t^{14}$	$2t^4+4t^6+7t^8+4t^{10}+3t^{12}+t^{14}+t^{16}$	$6t^6+9t^8+10t^{10}+6t^{12}+4t^{14}+2t^{16}+t^{18}$
$\varpi_2 + \varpi_8$	$t^2+3t^4+2t^6+t^8$	$t^2+3t^4+3t^6+2t^8+t^{10}$	$t^4+2t^6+t^8+t^{10}$	$3t^4+3t^6+3t^8+t^{10}+t^{12}$	t^8	$2t^4+4t^6+3t^8+2t^{10}+t^{12}$	$2t^4+5t^6+4t^8+3t^{10}+t^{12}+t^{14}$	$t^4+6t^6+8t^8+6t^{10}+4t^{12}+2t^{14}+t^{16}$
$\varpi_1 + 2\varpi_7$	$t^2+2t^4+2t^6+t^8$	$2t^4+3t^6+2t^8+t^{10}$	$t^2+t^4+2t^6+t^8+t^{10}$	$2t^4+3t^6+3t^8+t^{10}+t^{12}$	t^4+t^8	$t^4+3t^6+3t^8+2t^{10}+t^{12}$	$4t^6+4t^8+3t^{10}+t^{12}+t^{14}$	$2t^4+4t^6+8t^8+6t^{10}+4t^{12}+2t^{14}+t^{16}$
$\varpi_1 + \varpi_6$	$t^2+2t^4+t^6$	$t^2+3t^4+2t^6+t^8$	$t^4+t^6+t^8$	$t^2+2t^4+3t^6+t^8+t^{10}$	t^6	$2t^4+3t^6+2t^8+t^{10}$	$3t^4+4t^6+3t^8+t^{10}+t^{12}$	$2t^4+6t^6+6t^8+4t^{10}+2t^{12}+t^{14}$
$2\varpi_2$	$t^2+t^4+t^6$	$t^4+2t^6+t^8$	t^4+t^8	$2t^6+t^8+t^{10}$	t^6	$t^2+2t^4+2t^6+2t^8+t^{10}$	$2t^4+2t^6+3t^8+t^{10}+t^{12}$	$t^4+4t^6+4t^8+4t^{10}+2t^{12}+t^{14}$
$\varpi_7 + \varpi_8$	t^2+t^4	$t^2+t^4+t^6$	$t^2+t^4+t^6$	$t^2+2t^4+t^6+t^8$	0	$t^4+t^6+t^8$	$2t^4+2t^6+t^8+t^{10}$	$2t^4+4t^6+3t^8+2t^{10}+t^{12}$
$\varpi_1 + \varpi_3$	t^2+t^4	$t^2+2t^4+t^6$	t^6	$2t^4+t^6+t^8$	t^4	$t^2+2t^4+2t^6+t^8$	$t^2+2t^4+3t^6+t^8+t^{10}$	$3t^4+4t^6+4t^8+2t^{10}+t^{12}$
ϖ_4	t^2	t^2+t^4	t^4	$t^2+t^4+t^6$	0	t^4+t^6	$t^2+2t^4+t^6+t^8$	$2t^4+3t^6+2t^8+t^{10}$
$2\varpi_1 + \varpi_7$	t^2	t^2+t^4	t^4	t^4+t^6	t^2	$t^2+2t^4+t^6$	$2t^4+t^6+t^8$	$t^2+2t^4+4t^6+2t^8+t^{10}$
$\varpi_2 + \varpi_7$	1	t^2	t^2	t^2+t^4	0	t^2+t^4	$t^2+t^4+t^6$	$t^2+3t^4+2t^6+t^8$
$\varpi_1 + \varpi_8$	0	1	0	t^2	0	t^2	t^2+t^4	$t^2+2t^4+t^6$
$2\varpi_7$	0	0	1	t^2	0	0	t^4	$t^2+t^4+t^6$
ϖ_6	0	0	0	1	0	0	t^2	t^2+t^4
$3\varpi_1$	0	0	0	0	1	t^2+t^4	t^6	$t^4+t^6+t^8$
$\varpi_1 + \varpi_2$	0	0	0	0	0	1	t^2	t^2+t^4
ϖ_3	0	0	0	0	0	0	1	t^2
$\varpi_1 + \varpi_7$	0	0	0	0	0	0	0	1

	ϖ_8	$2\varpi_1$	ϖ_2	ϖ_7	ϖ_1	0
ϖ_5	$t^6+5t^8+8t^{10}+7t^{12}+$ $6t^{14}+4t^{16}+2t^{18}+t^{20}+$ t^{22}	$5t^{10}+4t^{12}+6t^{14}+3t^{16}+$ $3t^{18}+t^{20}+t^{22}$	$3t^8+6t^{10}+11t^{12}+8t^{14}+$ $7t^{16}+4t^{18}+3t^{20}+t^{22}+$ t^{24}	$5t^{10}+6t^{12}+9t^{14}+6t^{16}+$ $6t^{18}+3t^{20}+2t^{22}+t^{24}+$ t^{26}	$4t^{12}+5t^{14}+8t^{16}+5t^{18}+$ $5t^{20}+3t^{22}+2t^{24}+t^{26}+$ t^{28}	$t^{14}+3t^{18}+t^{20}+2t^{22}+$ $t^{24}+t^{26}+t^{30}$
$\varpi_3 + \varpi_7$	$2t^6+7t^8+7t^{10}+6t^{12}+$ $4t^{14}+2t^{16}+t^{18}+t^{20}$	$4t^8+4t^{10}+6t^{12}+3t^{14}+$ $3t^{16}+t^{18}+t^{20}$	$t^6+4t^8+10t^{10}+8t^{12}+$ $7t^{14}+4t^{16}+3t^{18}+t^{20}+$ t^{22}	$3t^8+5t^{10}+9t^{12}+6t^{14}+$ $6t^{16}+3t^{18}+2t^{20}+t^{22}+$ t^{24}	$3t^{10}+4t^{12}+8t^{14}+5t^{16}+$ $5t^{18}+3t^{20}+2t^{22}+t^{24}+$ t^{26}	$t^{12}+3t^{16}+t^{18}+2t^{20}+$ $t^{22}+t^{24}+t^{28}$
$\varpi_2 + \varpi_8$	$4t^6+5t^8+6t^{10}+4t^{12}+$ $2t^{14}+t^{16}+t^{18}$	$t^6+3t^8+5t^{10}+3t^{12}+$ $3t^{14}+t^{16}+t^{18}$	$t^6+7t^8+7t^{10}+7t^{12}+$ $4t^{14}+3t^{16}+t^{18}+t^{20}$	$3t^8+6t^{10}+6t^{12}+6t^{14}+$ $3t^{16}+2t^{18}+t^{20}+t^{22}$	$3t^{10}+6t^{12}+5t^{14}+5t^{16}+$ $3t^{18}+2t^{20}+t^{22}+t^{24}$	$2t^{14}+t^{16}+2t^{18}+t^{20}+$ $t^{22}+t^{26}$
$\varpi_1 + 2\varpi_7$	$2t^6+5t^8+6t^{10}+4t^{12}+$ $2t^{14}+t^{16}+t^{18}$	$2t^6+2t^8+5t^{10}+3t^{12}+$ $3t^{14}+t^{16}+t^{18}$	$t^6+5t^8+7t^{10}+7t^{12}+$ $4t^{14}+3t^{16}+t^{18}+t^{20}$	$t^6+2t^8+6t^{10}+6t^{12}+$ $6t^{14}+3t^{16}+2t^{18}+t^{20}+$ t^{22}	$2t^8+2t^{10}+6t^{12}+5t^{14}+$ $5t^{16}+3t^{18}+2t^{20}+t^{22}+$ t^{24}	$t^{10}+2t^{14}+t^{16}+2t^{18}+$ $t^{20}+t^{22}+t^{26}$
$\varpi_1 + \varpi_6$	$t^4+4t^6+6t^8+4t^{10}+$ $2t^{12}+t^{14}+t^{16}$	$t^6+4t^8+3t^{10}+3t^{12}+$ $t^{14}+t^{16}$	$3t^6+6t^8+7t^{10}+4t^{12}+$ $3t^{14}+t^{16}+t^{18}$	$t^6+4t^8+6t^{10}+6t^{12}+$ $3t^{14}+2t^{16}+t^{18}+t^{20}$	$t^8+4t^{10}+5t^{12}+5t^{14}+$ $3t^{16}+2t^{18}+t^{20}+t^{22}$	$t^{12}+t^{14}+2t^{16}+t^{18}+t^{20}+$ t^{24}
$2\varpi_2$	$t^6+4t^8+4t^{10}+2t^{12}+$ $t^{14}+t^{16}$	$t^4+t^6+4t^8+2t^{10}+3t^{12}+$ $t^{14}+t^{16}$	$3t^6+3t^8+6t^{10}+4t^{12}+$ $3t^{14}+t^{16}+t^{18}$	$3t^8+3t^{10}+6t^{12}+3t^{14}+$ $2t^{16}+t^{18}+t^{20}$	$t^8+4t^{10}+3t^{12}+5t^{14}+$ $3t^{16}+2t^{18}+t^{20}+t^{22}$	$2t^{12}+2t^{16}+t^{18}+t^{20}+t^{24}$
$\varpi_7 + \varpi_8$	$t^4+3t^6+3t^8+2t^{10}+t^{12}+$ t^{14}	$t^6+2t^8+2t^{10}+t^{12}+t^{14}$	$2t^6+4t^8+3t^{10}+3t^{12}+$ $t^{14}+t^{16}$	$t^6+3t^8+4t^{10}+3t^{12}+$ $2t^{14}+t^{16}+t^{18}$	$t^8+3t^{10}+3t^{12}+3t^{14}+$ $2t^{16}+t^{18}+t^{20}$	$t^{12}+t^{14}+t^{16}+t^{18}+t^{22}$
$\varpi_1 + \varpi_3$	$t^4+4t^6+4t^8+2t^{10}+t^{12}+$ t^{14}	$3t^6+2t^8+3t^{10}+t^{12}+t^{14}$	$t^4+3t^6+6t^8+4t^{10}+$ $3t^{12}+t^{14}+t^{16}$	$2t^6+3t^8+6t^{10}+3t^{12}+$ $2t^{14}+t^{16}+t^{18}$	$3t^8+3t^{10}+5t^{12}+3t^{14}+$ $2t^{16}+t^{18}+t^{20}$	$t^{10}+2t^{14}+t^{16}+t^{18}+t^{22}$
ϖ_4	$2t^4+3t^6+2t^8+t^{10}+t^{12}$	$t^6+2t^8+t^{10}+t^{12}$	$3t^6+3t^8+3t^{10}+t^{12}+t^{14}$	$t^6+4t^8+3t^{10}+2t^{12}+$ $t^{14}+t^{16}$	$t^8+3t^{10}+3t^{12}+2t^{14}+$ $t^{16}+t^{18}$	$t^{12}+t^{14}+t^{16}+t^{20}$
$2\varpi_1 + \varpi_7$	$t^4+3t^6+2t^8+t^{10}+t^{12}$	$2t^4+2t^6+3t^8+t^{10}+t^{12}$	$t^4+4t^6+4t^8+3t^{10}+t^{12}+$ t^{14}	$t^4+t^6+5t^8+3t^{10}+2t^{12}+$ $t^{14}+t^{16}$	$2t^6+2t^8+5t^{10}+3t^{12}+$ $2t^{14}+t^{16}+t^{18}$	$t^8+2t^{12}+t^{14}+t^{16}+t^{20}$
$\varpi_2 + \varpi_7$	$2t^4+2t^6+t^8+t^{10}$	$t^4+2t^6+t^8+t^{10}$	$2t^4+3t^6+3t^8+t^{10}+t^{12}$	$3t^6+3t^8+2t^{10}+t^{12}+t^{14}$	$t^6+3t^8+3t^{10}+2t^{12}+$ $t^{14}+t^{16}$	$t^{10}+t^{12}+t^{14}+t^{18}$
$\varpi_1 + \varpi_8$	$t^2+2t^4+t^6+t^8$	$t^4+t^6+t^8$	$2t^4+3t^6+t^8+t^{10}$	$t^4+3t^6+2t^8+t^{10}+t^{12}$	$t^6+3t^8+2t^{10}+t^{12}+t^{14}$	$t^{10}+t^{12}+t^{16}$
$2\varpi_7$	$t^4+t^6+t^8$	t^4+t^8	$2t^6+t^8+t^{10}$	$t^4+t^6+2t^8+t^{10}+t^{12}$	$t^6+t^8+2t^{10}+t^{12}+t^{14}$	$t^8+t^{12}+t^{16}$
ϖ_6	$t^2+t^4+t^6$	t^6	$2t^4+t^6+t^8$	$t^4+2t^6+t^8+t^{10}$	$t^6+2t^8+t^{10}+t^{12}$	$t^{10}+t^{14}$
$3\varpi_1$	t^8+t^{10}	$t^2+t^4+2t^6+t^8+t^{10}$	$t^4+2t^6+2t^8+t^{10}+t^{12}$	$t^6+t^8+2t^{10}+t^{12}+t^{14}$	$t^4+t^6+3t^8+2t^{10}+2t^{12}+$ $t^{14}+t^{16}$	$t^6+t^{10}+t^{12}+t^{14}+t^{18}$
$\varpi_1 + \varpi_2$	t^4+t^6	$t^2+t^4+t^6$	$t^2+2t^4+t^6+t^8$	$t^4+2t^6+t^8+t^{10}$	$t^4+2t^6+2t^8+t^{10}+t^{12}$	$t^8+t^{10}+t^{14}$
ϖ_3	t^2+t^4	t^4	$t^2+t^4+t^6$	$2t^4+t^6+t^8$	$2t^6+t^8+t^{10}$	t^8+t^{12}
$\varpi_1 + \varpi_7$	t^2	t^2	t^2+t^4	$t^2+t^4+t^6$	$2t^4+t^6+t^8$	t^6+t^{10}

TABLE 3.

	ϖ_8	$2\varpi_1$	ϖ_2	ϖ_7	ϖ_1	0
ϖ_8	1	0	t^2	t^2+t^4	t^4+t^6	t^8
$2\varpi_1$	0	1	t^2	t^4	$t^2+t^4+t^6$	t^4+t^8
ϖ_2	0	0	1	t^2	t^2+t^4	t^6
ϖ_7	0	0	0	1	t^2	t^4
ϖ_1	0	0	0	0	1	t^2
0	0	0	0	0	0	1

TABLE 4.

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